COMPRESSION AND BENDING STIFFNESS OF FIBER-REINFORCED ELASTOMERIC BEARINGS

Hsiang-Chuan Tsai, National Taiwan University of Science and Technology, Taipei, Taiwan

James M. Kelly, University of California, Berkeley, CA

Abstract

The reinforcing elements of multi-layer elastomeric isolation bearings, which are normally steel plates, are replaced by a fiber reinforcement. The fiber-reinforced isolator is significantly lighter and could lead to a much less labor-intensive manufacturing process. In contrast to the steel reinforcement, which is assumed to be rigid, the fiber reinforcement is flexible in extension. This paper presents theoretical approach for analyzing the compressive stiffness and bending stiffness of fiber-reinforced isolators. The elastomer is assumed to be incompressible and pressure-dominant. Assuming that each elastomeric layer in the bearing deforms in such a way that the horizontal planes remain planar and points on a vertical line lie on parabola after loading, the closed-form solutions are derived for the compression and bending stiffnesses of fiber-reinforced isolators having three types of geometry: infinitely long strip, rectangular and circular. The influence of fiber flexibility on the stiffness of isolators is studied.

Introduction

The bearings used in seismic isolation are heavy and expensive. The primary weight in an isolator is due to the reinforcing steel plates, which are used to provide vertical stiffness to the rubber-steel composite element. The high cost of producing the isolators results from the labor involved in preparing the steel plates and assembly of the rubber sheets and steel plates for vulcanization bonding in a mold. The research work recently performed by Kelly (1999) suggests that both the weight and the cost of isolators can be significantly reduced by eliminating the steel reinforcing plates and replacing them with a fiber reinforcement. The reduction in weight is possible because fiber materials are now available with an elastic stiffness that is of the same order as steel. The reinforcement needed to provide the vertical stiffness may be obtained by using a similar volume of a very much lighter material. Manufacturing cost may be reduced if the use of fiber allows a simpler, less labor-intensive process.

To calculate the stiffness of a steel-reinforced bearing, an approximate analysis is used that assumes that each individual layer of elastomer in the bearing deforms in such a way that horizontal planes remain planar and points on a vertical line lie on a parabola after loading. The steel plates are assumed to be rigid to constrain the displacement at the top and bottom of the elastomer. The elastomer is assumed to be strictly incompressible and its normal stress components are approximated by the pressure. This leads to the well-known ‘pressure solution’ approach (Kelly 1997). This paper will show that the extensional flexibility of the fiber reinforcement can be incorporated into the ‘pressure solution’ approach, and that prediction of the resulting stiffness can be made.

Governing Equation of Pressure

A single elastomeric layer with fiber reinforcements in an isolator is shown in Figure 1 where a coordinate system (x, y, z) is established by locating the origin at the center of the elastomeric layer and

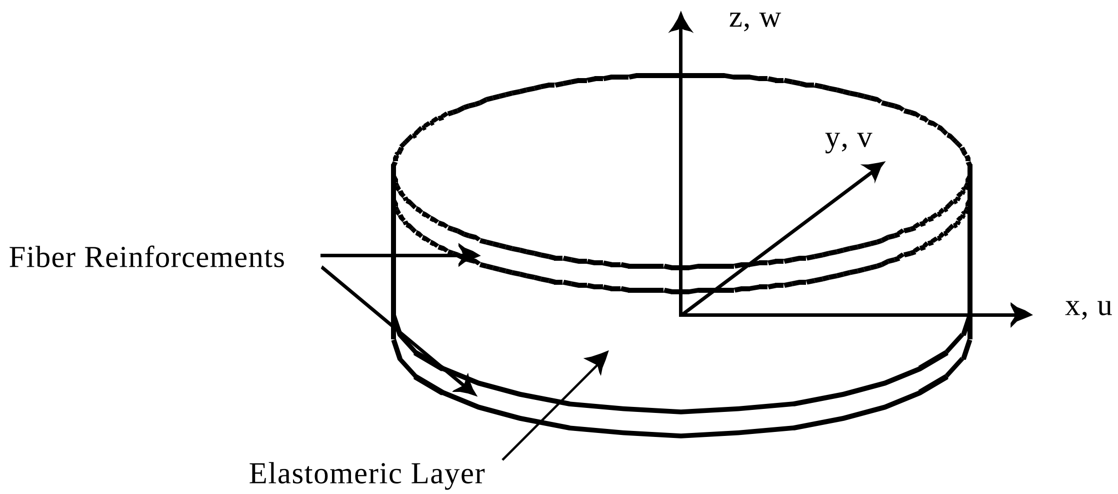

Figure 1. Elastomeric layer with fiber reinforcements

the x-y plane in the middle plane of the layer. The elastomeric layer has a thickness of t. Its top and bottom surfaces are perfectly bonded to fiber reinforcements which are modeled as an equivalent sheet of thickness t_f. Let u, v and w denote the displacements of the elastomer in the x, y and z coordinate directions, respectively; u_1 and v_1 denote the displacements of the reinforcement in the x and y directions, respectively. Under the compression load P in the z direction, the displacements of the elastomer are assumed to have the form

$$u(x, y, z) = u_0(x, y)\left(1 - \frac{4z^2}{t^2}\right) + u_1(x, y) \tag{1}$$

$$v(x, y, z) = v_0(x, y)\left(1 - \frac{4z^2}{t^2}\right) + v_1(x, y) \tag{2}$$

$$w(x, y, z) = w(z) \tag{3}$$

In Eqs. (1) and (2), the terms of u_0 and v_0 represent the kinematic assumption of quadratically varied displacements and are supplemented by additional displacements u_1 and v_1, respectively, which are constant through the thickness and are intended to accommodate the stretch of the reinforcement. Eq. (3) represents the assumption that horizontal planes remain planar.

The elastomer is assumed to have linearly elastic behavior with incompressibility. The assumption of incompressibility produces a constraint on displacements in the form

$$u_{,x} + v_{,y} + w_{,z} = 0 \tag{4}$$

Substituting Eqs. (1) to (3) into the above equation and then taking integration through the thickness from $z = -t/2$ to $z = t/2$ lead to

$$\frac{2}{3}(u_{0,x} + v_{0,y}) + u_{1,x} + v_{1,y} = \varepsilon_c \tag{5}$$

in which $\varepsilon_c = (w(-t/2) - w(t/2))/t$ is the nominal compression strain.

The stress state in the elastomer is assumed to be dominated by the internal pressure p, such that the stress components of the elastomer are (Kelly 1997)

$$\sigma_{xx} \approx \sigma_{yy} \approx \sigma_{zz} \approx -p; \qquad \sigma_{xy} \approx 0 \tag{6}$$

The equilibrium equations in the x and y directions for the stresses of the elastomer are then reduced to

$$-p_{,x} + \sigma_{xz,z} = 0 \tag{7}$$

$$-p_{,y} + \sigma_{yz,z} = 0 \tag{8}$$

Under the displacement assumptions in Eqs. (1) to (3), the shear stress components of the elastomer are

$$\sigma_{xz} = -\frac{8G}{t^2} z u_{0;} \qquad \sigma_{yz} = -\frac{8G}{t^2} z v_0 \tag{9}$$

with G being the shear modulus of the elastomer, and the equilibrium equations become

$$p_{,x} = -\frac{8G}{t^2} u_0 \tag{10}$$

$$p_{,y} = -\frac{8G}{t^2} v_0 \tag{11}$$

Differentiating Eqs. (10) and (11) with respect to x and y, respectively, and then adding them up yield

$$p_{,xx} + p_{,yy} = -\frac{8G}{t^2}(u_{0,x} + v_{0,y}) \tag{12}$$

The internal forces acting in the reinforcing sheet are related to the shear stresses, σ_{xz} and σ_{yz}, acting on the surfaces of the reinforcing sheet bonded to the top and bottom layers of elastomer through two equilibrium equations in the x and y directions

$$N_{xx,x} + N_{xy,y} + \sigma_{xz}\big|_{z=-t/2} - \sigma_{xz}\big|_{z=t/2} = 0 \tag{13}$$

$$N_{yy,y} + N_{xy,x} + \sigma_{yz}\big|_{z=-t/2} - \sigma_{yz}\big|_{z=t/2} = 0 \tag{14}$$

where N_{xx} and N_{yy} are the normal forces per unit length in the x and y directions, respectively; N_{xy} is the in-plane shear force per unit length. Substituting Eq. (9) into the above equations, and then combining these with the equilibrium equations of the elastomeric layer in Eqs. (10) and (11) to eliminate u_0 and v_0 give

$$N_{xx,x} + N_{xy,y} = tp_{,x} \tag{15}$$

$$N_{yy,y} + N_{xy,x} = tp_{,y} \tag{16}$$

Bringing the strain-stress relation of the reinforcement into Eqs. (15) and (16) leads to

$$\frac{E_f t_f}{1-\upsilon^2}(u_{1,xx} + \upsilon v_{1,yx}) + \frac{E_f t_f}{2(1+\upsilon)}(u_{1,yy} + v_{1,xy}) = tp_{,x} \tag{17}$$

$$\frac{E_f t_f}{1-\upsilon^2}(v_{1,yy} + \upsilon u_{1,xy}) + \frac{E_f t_f}{2(1+\upsilon)}(v_{1,xx} + u_{1,yx}) = tp_{,y} \tag{18}$$

where E_f and υ are the elastic modulus and Poisson's ratio of the reinforcement. Differentiating Eqs. (17) and (18) with respect to x and y, respectively, and then adding them up yield

$$(u_{1,x}+v_{1,y})_{,xx}+(u_{1,x}+v_{1,y})_{,yy} = \frac{t(1-\upsilon^2)}{E_f t_f}(p_{,xx}+p_{,yy}) \tag{19}$$

Combining Eq. (5) with Eq. (12) to eliminate the terms of u_0 and v_0 gives

$$u_{1,x}+v_{1,y} = \varepsilon_c + \frac{t^2}{12G}(p_{,xx}+p_{,yy}) \tag{20}$$

Substitution of this into Eq. (19) leads to

$$p_{,xxxx}+2p_{,xxyy}+p_{,yyyy}-\alpha^2(p_{,xx}+p_{,yy}) = 0 \tag{21}$$

in which α is defined as

$$\alpha = \sqrt{\frac{12G(1-\upsilon^2)}{E_f t_f t}} \tag{22}$$

The pressure p can be solved by satisfying the boundary conditions that the pressure in the elastomer and the normal force in the reinforcement vanish at the edges of the pad.

Effective Compressive Modulus

The compression stiffness of a bearing is determined by the effective compressive modulus E_c defined as

$$E_c = \frac{P}{A\varepsilon_c} \tag{23}$$

where A is the area of the pad in the x-y plane and the resultant compression load P has the form

$$P = -\int_A \sigma_{zz}\,dA \approx \int_A p(x,y)\,dA \tag{24}$$

For infinitely long strip pads, the deformation is in a plane strain state, so that displacement component in the y direction vanishes and the governing equation of the pressure becomes a ordinary differential equation of x. If the strip pad has a width of $2a$, the effective compressive modulus has the form

$$E_c = GS^2\frac{12}{(\alpha a)^2}\left(1-\frac{\tanh\alpha a}{\alpha a}\right) \tag{25}$$

in which $S = a/t$ is the shape factor of the infinitely long strip pad.

For rectangular pads, the pressure can be solved by using the approximate boundary conditions (Tsai and Kelly 2001). If the aspect of the rectangular pad is $2a$ by $2b$, the effective compressive modulus has the form

$$E_c = GS^2\frac{24}{\pi^2(\alpha a)^2}\left(1+\frac{a}{b}\right)^2\sum_{n=1}^{\infty}\frac{1}{\left(n-\frac{1}{2}\right)^2}\left(\frac{\tanh\gamma_n b}{\gamma_n b}-\frac{\tanh\beta_n b}{\beta_n b}+\frac{\tanh\bar{\gamma}_n a}{\bar{\gamma}_n a}-\frac{\tanh\bar{\beta}_n a}{\bar{\beta}_n a}\right) \tag{26}$$

in which $S = \dfrac{ab}{t(a+b)}$ is the shape factor of the rectangular pad, and

$$\gamma_n = \left(n-\frac{1}{2}\right)\frac{\pi}{a}; \qquad \beta_n = \sqrt{\gamma_n^2+\alpha^2}; \qquad \bar{\gamma}_n = \left(n-\frac{1}{2}\right)\frac{\pi}{b}; \qquad \bar{\beta}_n = \sqrt{\bar{\gamma}_n^2+\alpha^2} \tag{27}$$

The effective compressive modulus in Eq. (26) varies almost linearly with the aspect ratio a/b, so that a simplified formula is established (Tsai and Kelly 2001)

$$
\begin{aligned}
E_c &= GS^2 \frac{12}{(\alpha a)^2}\left(1 - \frac{\tanh \alpha a}{\alpha a}\right) \\
&\quad \left\{1 + \frac{a}{b}[-0.59 + 0.026(\alpha a) + 0.074(\alpha a)^2 - 0.022(\alpha a)^3 + 0.0019(\alpha a)^4]\right\}
\end{aligned}
\tag{28}
$$

with $a/b \le 1$. Because the range of the αa values used in the regression analysis is between 0 and 5, the effective compressive modulus in Eq. (28) is only applicable to the range of $0 \le \alpha a \le 5$. The maximum error in this range is smaller than 4 percent.

For circular pads under compression, the deformation is in an axisymmetric state. The governing equation of the pressure can be expressed by the cylindrical coordinate system (r, z). If the circular pad has a radius of a, the effective compressive modulus has the form

$$
E_c = GS^2 \frac{24(1 + \upsilon)}{(\alpha a)^2}\left[\frac{\alpha a I_0(\alpha a) - 2I_1(\alpha a)}{\alpha a I_0(\alpha a) - (1 - \upsilon)I_1(\alpha a)}\right] \tag{29}
$$

in which I_n is the modified Bessel function of the first kind of order n and $S = a/(2t)$ is the shape factor of the circular pad.

The ratio $E_c/(GS^2)$ is plotted in Figure 2 as a function of αa for the infinite long strip pad, the

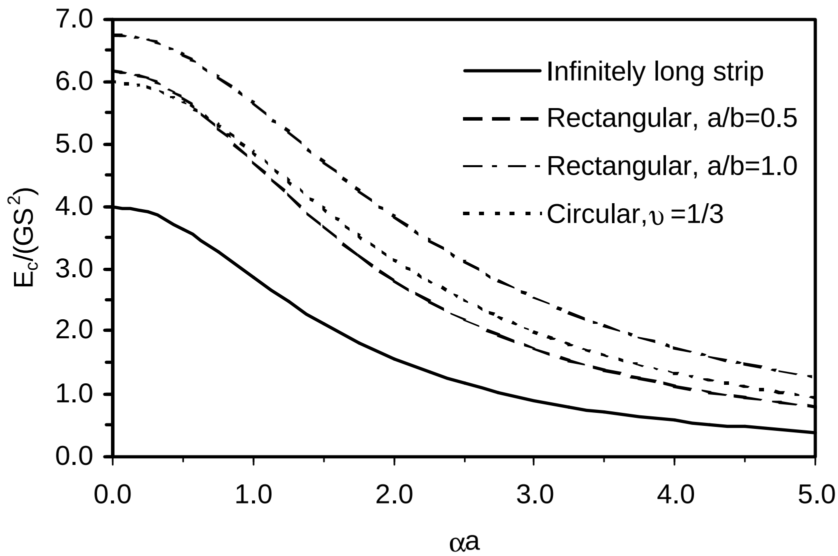

Figure 2. Variation of effective compressive modulus with αa

rectangular pads with $a/b = 0.5$ and $a/b = 1.0$, and the circular pad of $\upsilon = 1/3$. The figure shows that the effective compressive modulus decreases with increasing αa. To have high effective compressive modulus, we must keep the value of αa as low as possible.

Effective Bending Modulus

When the pad in Figure 1. is subjected to a pure bending moment M in the y direction, the reinforcements bonded to the top and bottom of the elastomeric layer rotate about the y axes. Assuming the reinforcements remain planar, the rotation forms an angle ϕ between the reinforcing sheets and is symmetric to the x-y plane. Following the same kinematic assumptions used for the compression stiffness, the displacement components of the elastomer in the x and y directions has the same forms as Eqs. (1) and (2), respectively. The displacement in the z direction is given by

$$w(x, y, z) = \frac{1}{\rho}xz \tag{30}$$

in which $\rho = t/\phi$ is the radius of the bending curvature. Following the similar procedure described in Section 2, the pressure governing equation for the bending stiffness can be derived and have the same form as Eq. (21).

The bending stiffness of the bearing is determined by the effective bending modulus E_b defined as

$$E_b = \frac{\rho M}{I_y} \tag{31}$$

where I_y is the moment of inertia of the pad about the y axis. The bending moment M is expressed as

$$M = \int_A \sigma_{zz} x dA \approx -\int_A p(x, y) x dA \tag{32}$$

For the infinitely long strip pad of width a, the moment of inertia is $I_y = (2/3)a^3$, and the effective bending modulus of the infinitely long strip pad has the form

$$E_b = GS^2 \frac{36}{(\alpha a)^4}\left[1 + \frac{1}{3}(\alpha a)^2 - \frac{\alpha a}{\tanh \alpha a}\right] \tag{33}$$

If the rectangular pad has a side length $2a$ along the x axis and $2b$ along the y axis, the moment of inertia is $I_y = (4/3)a^3 b$. Using the approximate boundary conditions (Tsai and Kelly 2001), the effective bending modulus of the rectangular pad can be solved as

$$E_b = GS^2 \frac{72}{\pi^2(\alpha a)^2}\left(1 + \frac{a}{b}\right)^2 \sum_{n=1}^{\infty}\left\{\frac{1}{n^2}\left(\frac{\tanh \tilde{\gamma}_n b}{\tilde{\gamma}_n b} - \frac{\tanh \tilde{\beta}_n b}{\tilde{\beta}_n b}\right) + \frac{1}{\left(n - \frac{1}{2}\right)^2}\left[\frac{1}{\bar{\gamma}_n a \tanh \bar{\gamma}_n a} - \frac{1}{(\bar{\gamma}_n a)^2} - \frac{1}{\bar{\beta}_n a \tanh \bar{\beta}_n a} + \frac{1}{(\bar{\beta}_n a)^2}\right]\right\} \tag{34}$$

in which $\bar{\gamma}_n$ and $\bar{\beta}_n$ have been defined in Eq. (27) and

$$\tilde{\gamma}_n = n\frac{\pi}{a}; \qquad \tilde{\beta}_n = \sqrt{\tilde{\gamma}_n^2 + \alpha^2} \tag{35}$$

Utilizing the regression analysis, a simplified formula for the effective bending modulus of rectangular pads is established (Tsai and Kelly 2001)

$$E_b = GS^2 \frac{36}{(\alpha a)^4}\left[1 + \frac{1}{3}(\alpha a)^2 - \frac{\alpha a}{\tanh \alpha a}\right] \left\{1 + \frac{a}{b}[-0.30 - 0.0024(\alpha a) + 0.021(\alpha a)^2 - 0.0045(\alpha a)^3 + 0.0003(\alpha a)^4]\right\} \tag{36}$$

with $a/b \le 1$. The errors of this simplified formula with respect to the exact formula in Eq. (34) is smaller than 0.6 percent. Because the range of the αa values used in the regression analysis is between 0 and 5, the effective bending modulus in Eq. (36) is only applicable to the range of $0 \le \alpha a \le 5$.

For the circular pad of radius a, the moment of inertia about the r axis is $I_r = \pi a^4/4$. The effective bending modulus of circular pads is solved as (Tsai and Kelly 2001)

$$E_b = GS^2 \frac{24(1+\upsilon)}{(\alpha a)^2}\left[\frac{\alpha a \mathrm{I}_1(\alpha a) - 4\mathrm{I}_2(\alpha a)}{\alpha a \mathrm{I}_1(\alpha a) - 2(1-\upsilon)\mathrm{I}_2(\alpha a)}\right] \tag{37}$$

The ratio $E_b/(GS^2)$ is plotted in Figure 3 as a function of αa for the infinite long strip pad, the

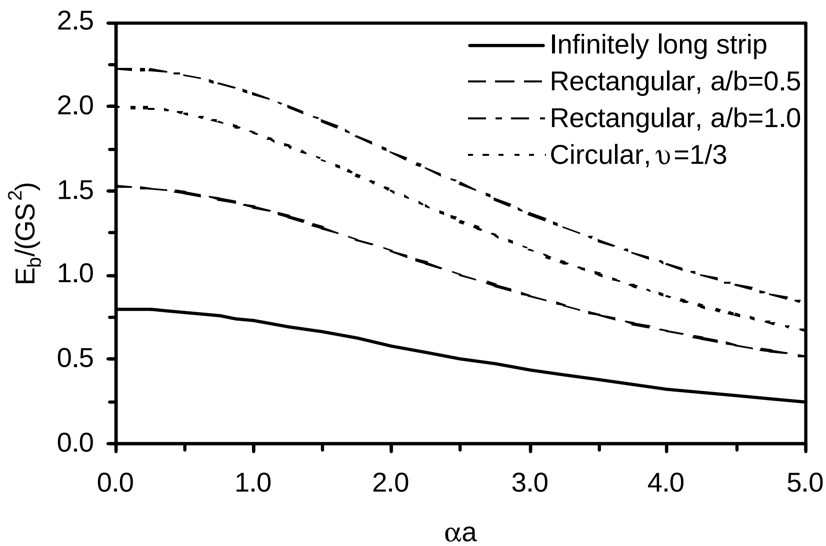

Figure 3. Variation of effective bending modulus with αa

rectangular pads with $a/b = 0.5$ and $a/b = 1.0$, and the circular pad of $\upsilon = 1/3$. The figure shows that the effective bending modulus decreases with increasing αa, but the variation is less severe than the effective compression modulus in Figure 2.

Conclusion

Theoretical analyses on different shapes of elastomeric layer bonded to flexible reinforcements and subjected to compression loading and pure bending loading are presented. Theoretical solutions show that the compression stiffness and the bending stiffness of the fiber-reinforced isolator are affected by the shape factor of the elastomer and the flexibility of the reinforcement. Similar to steel-reinforced isolators, the stiffness of fiber-reinforced isolators increases with increasing the shape factor, but the flexibility of the reinforcement can decrease the stiffness of the isolator.

References

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